

# On Generalized Weil Representations over Involutive Rings

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**ABSTRACT.** We construct via generators and relations, generalized Weil representations for analogues of classical  $SL(2, k)$ ,  $k$  a field, over involutive base rings  $(A, *)$ . This family of groups covers different kinds of groups, classical and non classical. We give some examples that include symplectic groups as well as non classical groups like  $SL_*(2, A_m)$ , where  $A_m$  is the finite modular analogue of the algebra of real m-jets in one dimension with its canonical involutive symmetry.

## 1. Introduction

The classification and construction of all unitary representations of a given group is a fundamental problem in representation theory. Especially interesting is the case of classical groups over a local field (particularly Lie groups) or over a finite field. It turns out sometimes that, remarkably enough, we can transpose methods of construction from Lie groups to finite groups of Lie type or vice versa.

So we may try to use finite groups of Lie type as a testing ground for designing methods of construction that could be transposed to the case of Lie groups. In principle, we may expect to be more feasible to find for the former, methods that are elementary, uniform and universal in the sense that they can be carried over to the latter.

This is however a hard task and one of the few finite groups of Lie type for which this has been achieved up to now, is  $G = SL(2, \mathbb{F}_q)$ , all of whose irreducible representations may be constructed elementary and uniformly by constructing a remarkable representation of  $G$  associated to each semi-simple two-dimensional (Galois) algebra over the finite field  $\mathbb{F}_q$  (namely  $\mathbb{F}_q \times \mathbb{F}_q$  and  $\mathbb{F}_{q^2}$ , up to isomorphism), which gives by decomposition according to the orthogonal group of the corresponding Galois norm, the principal and the cuspidal series of irreducible representations of  $G$ , respectively [11].

We recall that these representations are called nowadays Weil (or Shale-Weil) representations of  $G$ , because they arose from Weil's classical construction [13], for any local field, of the "oscillator representation" arising in the work of Shale [10] on linear symmetries of free boson fields in quantum field theory. Interestingly enough, this construction arising in quantum physics enabled us in the

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sixties to solve the fundamental problem for  $SL(2, \mathbb{F}_q)$ , had remained open since the construction of its character table at the turn of the century.

Originally Shale constructed the oscillator representation of  $Sp(2n, k)$ , for  $k = \mathbb{R}$ , which turns out to be a projective representation, taking advantage of the representation theory of the Heisenberg group  $H_n$  in  $n$  degrees of freedom, as described by the Stone von Neumann Theorem, that says that  $H_n$  has just one irreducible unitary representation of dimension greater than 1 with a given central character (the famous Schrödinger representation). Weil [13] extended later this construction to the local field case (for a very readable and more geometric account, see [6]). Shortly thereafter, Cartier noticed that Weil representations associated to any non-degenerate quadratic form  $Q$  over the base field, could be constructed for  $SL(2, k)$  in an elementary way, via generators and relations, using the well known presentation of this group, which follows from its Bruhat decomposition. In this way the Weil representation appears as a functorial construction on the category of quadratic spaces. The projective oscillator representation corresponds then to the rank one quadratic form  $x^2$  over  $k$  and we get in fact (true) Weil representations in the even rank case [11]. Moreover, under this approach the intertwining of the Weil representation  $W_Q$  associated to the quadratic form  $Q$  is “explained” by the natural action of the orthogonal group  $O(Q)$  in the space of  $W_Q$ . Cartier conjectured further that this construction would provide a uniform and elementary method for constructing all irreducible representations of  $SL(2, k, )$  and for  $Sp(2n, k)$  as well, although the presentation known at the time for  $Sp(2n, k)$  (found by Dickson at the turn of the century) was rather unyielding. A more convenient presentation based on the Bruhat decomposition, was found however in [11] by regarding  $Sp(2n, k)$  as an “ $SL(2)$ ” group over the involutive ring  $A = M(n, k)$  (with transpose as involution). An elementary construction of these Weil representations for the symplectic groups became then possible, via generators and relations.

This suggests to try to construct Weil representations by generators and relations in a very general setting, for an analogue  $G$  of  $SL(2, k)$  over any involutive ring, for which a “Bruhat” presentation analogue to the classical one holds. The method of construction would involve defining suitable “Weil operators” associated to the generators and checking that the defining relations of the presentation are preserved.

The central result of this paper is the construction of a very general Weil representation from abstract core data consisting of a module  $M$  over an involutive ring  $(A, *)$  equipped with a suitable non degenerate complex valued self pairing  $\chi$  and its second order homogeneous companion  $\gamma$ . We recover as particular examples of our construction, the Weil representations of the symplectic groups  $Sp(2n, k)$  of [11] and the generalized Weil representation in [5], for the non classical case of an involutive base ring having a nilpotent radical.

We also give below a first general decomposition of this generalized Weil representation, based on the symmetry group of our data.

We may remark that there is an intriguing similarity of our involutive analogues and quantum analogues of classical  $SL(2, k)$ , seen as matrix groups with non commuting entries, that could be “explained” as follows. We are working in fact in a tamely non commutative case, in which non commutativity is “controlled” by an involution  $T : a \mapsto a^*$  in the coefficient ring  $A$ . and we require our entries to  $*$ -commute. Indeed letting  $m : A \otimes A \rightarrow A$  the multiplication of  $A$  and  $S : A \otimes A \rightarrow A \otimes A$  the “flip”  $x \otimes y \mapsto y \otimes x$ , we see that  $T$  “transforms”  $m$  into  $m \circ S$  as follows:  $m \circ S = T \circ m \circ (T \otimes T)^{-1}$ , just as the  $R$ -matrix “controls” the lack of co-commutativity in a quantum group, i. e.  $S \circ \Delta = \mathcal{I}_R \circ \Delta$ , where  $\Delta$  stands for the comultiplication in the corresponding Hopf algebra and  $\mathcal{I}_R$  denotes conjugation by the  $R$ -matrix, which is an invertible element in

$A \otimes A$ . Our  $*$ -determinant  $ad^* - bc^*$ , defined only for matrices whose entries “ $*$ -commute” follows, in a way closely reminiscent of the  $q$ -determinant.

Regarding the existence of Bruhat presentations for involutive analogues of  $SL_*(2, A)$ , we mention that a first step was already accomplished in [8], where a *Bruhat decomposition* for these groups was obtained in the case of an artinian involutive base ring  $A$ . Later, the classical Bruhat presentation of  $SL(2, k)$  was extended to the case of an artinian simple involutive  $A$  in [7]. Then the existence of a Bruhat presentation for  $SL_*(2, A)$  was proved in [9] for a wide class of involutive rings  $A$ , namely, those admitting a weak  $*$ -analogue of the euclidean algorithm for coprime elements, that includes the artinian simple rings considered in [7] as well as rings as  $\mathbb{Z}$ . We notice that in this last paper,  $\varepsilon$ -analogues, ( $\varepsilon = \pm 1$ ) besides  $*$ -analogues, were considered for  $SL(2, k)$ , with  $\varepsilon = 1$  corresponding to the previous case, so that now we recover split orthogonal groups as well as symplectic groups.

This article is organized as follows.

In section 2 we recall the definition and main properties of the  $*$  and  $\varepsilon$  analogues of classical  $SL(2, k)$ .

In section 3, we recall Bruhat decompositions for our groups  $SL_*^\varepsilon(2, A)$ .

In section 4, we construct generalized Weil representations for  $SL_*^\varepsilon(2, A)$ .

In section 5, we show, for the case of symplectic groups, how to recover the Weil representations constructed in [11] and we construct also Weil representations for the case of an odd rank quadratic form, not considered there.

In section 6, we show how to recover the Weil representation constructed in the non classical case of an involutive ring with non trivial nilpotent Jacobson radical in [5].

Finally, in section 7, we give a first decomposition of our generalized Weil representations, in terms of the irreducible representations of the defining data  $\chi$  and  $\gamma$ .

## 2. The groups $SL_*^\varepsilon(2, A)$

We recall in this section the definition and main properties of the generalized classical groups  $SL_*^\varepsilon(2, A)$ , where  $(A, *)$  is a unitary ring with involution  $*$  and  $\varepsilon = \pm 1 \in A$ . We look upon these groups as  $*$  and  $\varepsilon$ -analogues of the groups  $SL(2, k)$ ,  $k$  a field. For more details see [8].

**Definition 2.1.** For a ring  $A$  with involution  $*$ , the group  $SL_*^\varepsilon(2, A)$  may be defined as the set of all  $2 \times 2$  matrices  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  with  $a, b, c, d \in A$  such that

- (1)  $ab^* = -\varepsilon ba^*$ ,
- (2)  $cd^* = -\varepsilon dc^*$ ,
- (3)  $a^*c = -\varepsilon c^*a$ ,
- (4)  $b^*d = -\varepsilon d^*b$ ,
- (5)  $ad^* + \varepsilon bc^* = a^*d + \varepsilon c^*b = 1$

with matrix multiplication.

We notice that this group may also be described as the unitary group  $U(H_\varepsilon)$  of the  $\varepsilon$ -Hermitian form  $H_\varepsilon$  on  $A^2 = A \times A$  associated to the  $2 \times 2$  matrix  $J_\varepsilon = \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix}$ , in the following sense.

**Definition 2.2.** A left  $\varepsilon$ -Hermitian form  $H$  on a left  $A$ -module  $M$  is a function  $H : M \times M \rightarrow A$  that is biadditive, left linear in the first variable and such that  $H(y, x) = \varepsilon H(x, y)^*$ , for all  $x, y \in M$ .

Recall that when the  $A$ -module  $M$  is free of finite rank  $m$ , we have a matrix description of left  $\varepsilon$ -Hermitian forms, as follows. We extend first the involution  $*$  in  $A$  to the full matrix ring  $M(m, A)$  of all  $m \times m$  matrices with coefficients in  $A$ , putting  $T^* = (t_{ji}^*)_{1 \leq i, j \leq m}$  for any  $T = (t_{ij})_{1 \leq i, j \leq m} \in M(m, A)$ . Now, a basis  $\mathcal{B} = \{e_1, \dots, e_m\}$  of the free  $A$ -module  $M$  having been chosen, we define the matrix  $[H]$  of  $H$  with respect to  $\mathcal{B}$  by  $[H] = (H(e_i, e_j))_{1 \leq i, j \leq m}$ .

Then  $H(u, v) = u[H]v^*$  ( $u, v \in M$ ) and conversely, given a matrix  $T$  such that  $T^* = \varepsilon T$  we recover an  $\varepsilon$ -Hermitian form  $H_T$  by  $H_T(u, v) = uTv^*$  ( $u, v \in M$ ). Notice that above we have written just  $H_\varepsilon$  instead of  $H_{J_\varepsilon}$  for the associated  $\varepsilon$ -hermitian form to the matrix  $J_\varepsilon$ .

**Proposition 2.3.** *The group  $SL_*^\varepsilon(2, A)$  may be defined equivalently as the set of all automorphisms  $g$  of the  $A$ -module  $M = A \times A$  such that  $H_\varepsilon \circ (g \times g) = H_\varepsilon$  or in matrix form as*

$$SL_*^\varepsilon(2, A) = \{T \in M(2, A) \mid TJ_\varepsilon T^* = J_\varepsilon\}.$$

□

### 3. Bruhat presentations of $SL_*^\varepsilon(2, A)$

Given a unitary ring  $A$  with an involution  $*$ , we will write  $A^{sym}$  to denote the set of all  $\varepsilon$ -symmetric elements in  $A$ , i. e., elements  $a \in A$  such that  $a^* = -\varepsilon a$ . We set

$$h_t = \begin{pmatrix} t & 0 \\ 0 & t^{*-1} \end{pmatrix} \quad (t \in A^\times), \quad w = w_\varepsilon = \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix} \quad \text{and} \quad u_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \quad (s \in A^{sym})$$

**Definition 3.1.** We will say that  $G = SL_*^\varepsilon(2, A)$  has a Bruhat presentation (see [9, 5]) if it is generated by the above elements with defining relations

- (1)  $h_t h_{t'} = h_{tt'}$ ,  $u_b u_{b'} = u_{b+b'}$ ;
- (2)  $w^2 = h_\varepsilon$ ;
- (3)  $h_t u_b = u_{tb} h_t$ ;
- (4)  $wh_t = h_{t^{*-1}} w$ ;
- (5)  $wu_{t^{-1}} w u_{-\varepsilon t} w u_{t^{-1}} = h_{-\varepsilon t}$ , with  $t$  an invertible  $\varepsilon$ -symmetric element in  $A$ .

**Remark 3.2.** The relations above are called “universal” in loc. cit. because they hold for every possible involutive base ring  $(A, *)$ .

### 4. A generalized Weil representation for $G = SL_*^\varepsilon(2, A)$ .

In this section  $A$  denotes a ring  $A$  with an involution  $*$ , i.e. an antiautomorphism of  $A$  of order 2. In what follows we will assume that the ring  $A$  is finite and also that the group  $G = SL_*^\varepsilon(2, A)$  has a Bruhat presentation.

**4.1. Data for constructing a Weil representation of  $G = SL_*^\varepsilon(2, A)$ .** We will construct a generalized Weil representation for  $G$ , associated to the following data:

- (1) A finite right  $A$ -module  $M$ .
- (2) A bi-additive function  $\chi : M \times M \rightarrow \mathbb{C}^\times$  and a character  $\alpha \in \widehat{A^\times}$  such that:
  - (a)  $\chi(xt, y) = \alpha(tt^*)\chi(x, yt^*)$  for  $x, y \in M$  and  $t \in A^\times$  ( $\chi$  is  $\alpha$ -balanced).
  - (b)  $\chi(y, x) = [\chi(x, y)]^{-\varepsilon}$ . We observe that  $[\chi(x, y)]^{-\varepsilon} = \chi(-\varepsilon x, y)$  ( $\chi$  is  $\varepsilon$ -symmetric).
  - (c)  $\chi(x, y) = 1$  for any  $x \in M$ , implies  $y = 0$  ( $\chi$  is non-degenerate).
- (3) A function  $\gamma : A^{sym} \times M \rightarrow \mathbb{C}^\times$  such that:

- (a)  $\gamma(b + b', x) = \gamma(b, x)\gamma(b', x)$ , for all  $b, b' \in A^{sym}$  and  $x \in M$ .
- (b)  $\gamma(b, xt) = \gamma(tbt^*, x)$  or equivalently  $\gamma(b, x) = \gamma(tbt^*, xt^{-1})$ , for all  $b \in A^{sym}$ ,  $t \in A^\times$  and  $x \in M$ .
- (c)  $\gamma(t, x + z) = \gamma(t, x)\gamma(t, z)\chi(x, zt)$  for all  $x, z \in M, t \in A^{sym}$ . where  $t$  is  $\varepsilon$ -symmetric invertible in  $A$  and  $c \in \mathbb{C}^\times$  satisfies  $c^2|M| = \alpha(\varepsilon)$ .
- (4) We assume moreover that these data are related by the equation:

$$(4.1) \quad c\gamma(-\varepsilon t, x) \sum_{y \in M} \chi(-\varepsilon x, y)\gamma(t^{-1}, y) = \alpha(-t).$$

**Lemma 4.1.** *If  $t$  is an  $\varepsilon$ -symmetric invertible element in  $A$ , then  $\gamma(-\varepsilon t, x) = \gamma(t^{-1}, xt)$ .*

PROOF. Since  $t$  is  $\varepsilon$ -symmetric we have  $t^* = -\varepsilon t$ . Hence using the second condition on  $\gamma$  we get  $\gamma(t^{-1}, xt) = \gamma(tt^{-1}t^*, x) = \gamma(-\varepsilon t, x)$  and our lemma follows.  $\square$

**Proposition 4.2.** *We consider as above  $\chi, \gamma, \alpha$  and  $c$  satisfying  $c^2|M| = \alpha(\varepsilon)$ . The following two relations are equivalent:*

- (1)  $c\gamma(-\varepsilon t, x) \sum_{y \in M} \chi(-\varepsilon x, y)\gamma(t^{-1}, y) = \alpha(-t)$ .
- (2)  $\sum_{y \in M} \gamma(t, y) = \frac{\alpha(\varepsilon t)}{c}$ .

PROOF. The relation  $c\gamma(-\varepsilon t, x) \sum_{y \in M} \chi(-\varepsilon x, y)\gamma(t^{-1}, y) = \alpha(-t)$  is equivalent to

$$\sum_{y \in M} \gamma(t^{-1}, y)\gamma(-\varepsilon t, x)\chi(-\varepsilon x, y) = \frac{\alpha(-t)}{c}.$$

Now, given that  $t^* = -\varepsilon t$  and that  $\gamma(-\varepsilon t, x) = \gamma(t^{-1}, xt)$ , we get that the last above relation is equivalent to

$\sum_{y \in M} \gamma(t^{-1}, y)\gamma(t^{-1}, xt)\chi(-\varepsilon x, y) = \frac{\alpha(-t)}{c}$ . Using the properties of  $\gamma$  this last expression is the same as

$$\sum_{y \in M} \frac{\gamma(t^{-1}, y+xt)}{\chi(y, x)}\chi(-\varepsilon x, y) = \frac{\alpha(-t)}{c}.$$

But given that  $\frac{\chi(-\varepsilon x, y)}{\chi(y, x)} = 1$ , we are reduced to

$$\sum_{y \in M} \gamma(t^{-1}, y + xt) = \frac{\alpha(-t)}{c}, \text{ from which the result follows.}$$

$\square$

**4.2. Construction of the generalized Weil representation of  $G$ .** In the theorem below we keep the notations and hypotheses introduced in section 4.1.

**Theorem 4.3.** *There is a representation  $(\mathbb{C}^M, \rho)$  of  $G$  such that*

$$\begin{aligned} \rho_{u_b}(e_x) &= \gamma(b, x)e_x \\ \rho_{h_t}(e_x) &= \alpha(t)e_{xt^{-1}} \\ \rho_w(e_x) &= c \sum_{y \in M} \chi(-\varepsilon x, y)e_y \end{aligned}$$

for  $x \in M, b \in A^\times \cap A^\times, t \in A^\times$ , where  $e_x$  denotes Dirac delta function at  $x \in M$  given by  $e_x(y) = 1$  if  $y = x$ ;  $e_x(y) = 0$  otherwise.

This representation is called the generalized Weil representation of  $SL_*^\varepsilon(2, A)$  associated to the data  $(M, \alpha, \gamma, \chi)$ .

PROOF. It is enough to verify that  $\rho_{u_b}, \rho_{h_t}, \rho_w$  defined as above, satisfy the relations corresponding to the universal relations in the Bruhat presentation of  $G$ .

To this end, we observe that

- (i)  $(\rho_{h_t} \circ \rho_{h_{t'}})(e_x) = \rho_{h_t}(\alpha(t')e_{xt'^{-1}}) = \alpha(t')\alpha(t)e_{xt'^{-1}t^{-1}} = \alpha(t')\alpha(t)e_{x(tt')^{-1}}$   
 $= \alpha(tt')e_{x(tt')^{-1}} = \rho_{h_{tt'}}(e_x)$
- (ii)  $(\rho_{u_b} \circ \rho_{u_{b'}})(e_x) = \gamma(b', x)\gamma(b, x)e_x = \gamma(b + b', x)e_x = (\rho_{u_{b+b'}})(e_x)$
- (iii)  $(\rho_w \circ \rho_w)(e_x) = c^2 \sum_{z \in M} \sum_{y \in M} \chi(-\varepsilon x, y)\chi(-\varepsilon y, z)e_z = \alpha(\varepsilon)e_{x\varepsilon} = \rho_{h_\varepsilon}e_x$
- (iv)  $(\rho_{h_t} \circ \rho_{u_b})(e_x) = \gamma(b, x)\alpha(t)e_{xt^{-1}} = \alpha(t)\gamma(tbt^*, xt^{-1})e_{xt^{-1}} = (\rho_{tbt^*} \circ \rho_{h_t})(e_x)$
- (v)  $\rho_w \circ \rho_{h_t}(e_x) = c\alpha(t)\sum_y \chi(-\varepsilon xt^{-1}, y)e_y = c\sum_y \chi(-\varepsilon x, yt^{*-1})\alpha(t^{*-1})e_y =$   
 $\rho_{h_{t^{*-1}}}(c\sum_y \chi(-\varepsilon x, y)e_y) = (\rho_{h_{t^{*-1}}} \circ \rho_w)(e_x)$

(vi) Finally, a computation shows that

$$(\rho_w \rho_{u_{t^{-1}}} \rho_w \rho_{u_{-\varepsilon t}})(e_x) = \sum_z c^2 \gamma(-\varepsilon t, x) \left( \sum_y \chi(-\varepsilon x, y)\gamma(t^{-1}, y)\chi(-\varepsilon y, z) \right) e_z$$

$$(\rho_{h_{-\varepsilon t}} \rho_{u_{t^{-1}}} \rho_w \rho_{h_\varepsilon})(e_x) = c\alpha(-t)\sum_z \chi(-\varepsilon x, -\varepsilon zt)\gamma(-t^{-1}, -\varepsilon zt)e_z.$$

So we want to prove that

$$c\gamma(-\varepsilon t, x)\sum_y \chi(-\varepsilon x, y)\gamma(t^{-1}, y)\chi(-\varepsilon y, z) = \alpha(-t)\chi(-\varepsilon x, -\varepsilon zt)\gamma(-t^{-1}, -\varepsilon zt)$$

But by hypothesis

$$c\gamma(-\varepsilon t, x - \varepsilon z)\sum_{y \in M} \chi(-\varepsilon(x - \varepsilon z), y)\gamma(-\varepsilon t, yt^{-1}) = \alpha(-t),$$

so

$$c\gamma(-\varepsilon t, x)\sum_y \chi(-\varepsilon x, y)\gamma(t^{-1}, y)\chi(-\varepsilon y, z) = c\gamma(-\varepsilon t, x)\sum_y \chi(-\varepsilon x + z, y)\gamma(-\varepsilon t, yt^{-1}) =$$

$$c\gamma(-\varepsilon t, x)\frac{\alpha(-t)}{c\gamma(-\varepsilon t, x - \varepsilon z)} = \frac{\gamma(-\varepsilon t, x)\alpha(-t)}{\gamma(-\varepsilon t, x)\gamma(-\varepsilon t, -\varepsilon z)\chi(x, zt)} = \alpha(-t)\chi(-\varepsilon x, -\varepsilon zt)\gamma(-t^{-1}, -\varepsilon zt),$$

from which the result.  $\square$

Notice finally that theorem 4.3 may be reworded in more functional analytic terms as follows.

**Theorem 4.4.** *Let  $M$  be a finite  $A$ -right module. Denote  $L^2(M)$  the vector space of all complex-valued functions on  $M$ , endowed with the usual  $L^2$  inner product. Set*

- (1)  $\rho(h_t)(f)(x) = \bar{\alpha}(t)f(xt)$ ,  $f \in L^2(M)$  and  $t \in A^\times$ ,  $x \in M$ .
- (2)  $\rho(u_b)(f)(x) = \gamma(b, x)f(x)$ ,  $f \in L^2(M)$  and  $b \in A^{sym}$ ,  $x \in M$ .
- (3)  $\rho(w)(f)(x) = c \sum_{y \in M} \chi(-\varepsilon x, y)f(y)$ ,  $f \in L^2(M)$  and  $x \in M$ .

(where  $\bar{\alpha}$  denotes the complex conjugate of the character  $\alpha$ ).

These formulas define a unitary linear representation  $(L^2(M), \rho)$  of  $SL_*^\varepsilon(2, A)$ , called the generalized Weil representation of  $SL_*^\varepsilon(2, A)$  associated to the data  $(M, \alpha, \gamma, \chi)$ .

PROOF. It is enough to verify that the linear operators  $\rho_{u_b}, \rho_{h_t}, \rho_w$ , defined as above, preserve the universal relations for the generators of  $G = SL_*^\varepsilon(2, A)$ . This has been done however in the previous theorem, by evaluating on the canonical Dirac delta function basis  $\{e_x\}_{x \in M}$  of  $L^2(M)$ . On the other hand, the unitarity of the given operators is easily checked.  $\square$

### 5. Example 1: The full matrix ring case.

In this section we take  $A$  to be the full matrix ring  $M_n(k)$ , where  $k$  denotes the finite field  $\mathbb{F}_q$  with  $q$  elements, endowed with the transpose involution  $*$ .

Recall that we have proved in [9] that the group  $SL_*^\varepsilon(2, A)$  has a Bruhat presentation when  $q > 3$ .

To satisfy the conditions in Theorem 4.4 we may take (keeping the notations introduced there)  $M$  to be the  $k$ -vector space  $\text{Hom}_k(E_0, E)$  where  $E_0 = k^n$  and  $E$  is a  $k$ -vector space of dimension  $m$  endowed with a non-degenerate  $k$ -quadratic form  $Q_0$  with associated  $k$ -bilinear form  $B_0$ , defined by

$$B_0(u, v) = Q_0(u + v) - Q_0(u) - Q_0(v)$$

for all  $u, v \in E$ . In what follows we will identify  $M = \text{Hom}_k(E_0, E)$  with  $\bigoplus_{1 \leq i \leq n} E_i$ , where  $E_i = E_0$  for all  $i$ .

This induces canonically a non degenerate  $A$ -valued quadratic form  $\mathbb{Q}$  on  $M$  given by

- (1)  $\mathbb{Q}(x)_{ii} = Q_0(x_i)$
- (2)  $\mathbb{Q}(x)_{ij} = B_0(x_i, x_j)$
- (3)  $\mathbb{Q}(x)_{ji} = 0$

for all  $x = (x_1, \dots, x_n) \in M, 1 \leq i, j \leq n$ .

If we pass now to the quotient modulo “anti-traces”, i.e. if we define  $\overline{\mathbb{Q}} = pr \circ \mathbb{Q} : M \rightarrow \bar{A}$ , where  $pr$  denotes the canonical projection of  $A$  onto  $A/A^0$ , with  $A^0 = \{a - a^* | a \in A\}$ , then we see that  $(M, \overline{\mathbb{Q}})$  is a quadratic module over the involutive ring  $A$  in the sense of Tits [12].

We fix moreover a non trivial character  $\psi$  of  $k^+$  and we denote by  $tr$  the usual matrix trace from  $A$  onto  $k$ . Then  $\underline{\psi} = \psi \circ tr$  is a non trivial character of  $A^+$  such that  $\underline{\psi}(ab) = \underline{\psi}(ba)$  and  $\underline{\psi}(a^*) = \underline{\psi}(a)$  for all  $a, b \in A$ . On the other hand, we have =

$$\underline{\psi} \circ \mathbb{Q} = \psi \circ Q$$

where  $Q$  denotes the  $k$ -valued non-degenerate quadratic form  $tr \circ \mathbb{Q}$  over  $k$ , whose associated  $k$ -bilinear form will be denote by  $B$ . Notice that  $rank Q = nm$  and in fact

$$Q(x) = \sum_{1 \leq i \leq n} Q_0(x_i)$$

for all  $x \in M$ .

In what follows we assume that  $\varepsilon = -1$ . We put then for  $s \in A^{sym}, x, y \in M$  :

- (1)  $\gamma(s, x) = \underline{\psi}(s\overline{\mathbb{Q}}(x))$
- (2)  $\chi(x, y) = \underline{\psi}(\overline{\mathbb{B}}(x, y))$ .

From our results in 3.3.2 of [11] it follows easily that the normalized quadratic Gauss sum

$$S_{\underline{\psi}, \mathbb{Q}}(s) = \frac{1}{|M|^{\frac{1}{2}}} \sum_{x \in M} \underline{\psi}(s\mathbb{Q}(x)) = S_{\underline{\psi}, s\mathbb{Q}}(1)$$

associated to the matrix valued quadratic form  $\mathbb{Q}$ , defined for any  $s \in A^{sym}$ , is constant on the orbits of the natural action of the multiplicative group  $A^\times$  in the set  $A^\times \cap A^{sym}$ , given by  $s \mapsto asa^*$  ( $s \in A^\times \cap A^{sym}$ ,  $a \in A^\times$ ). Recall however that there are only two  $A^\times$ -orbits in  $A^\times \cap A^{sym}$ , that correspond to the two isomorphy types of non-degenerate  $k$ -quadratic forms of rank  $n$ , that may be represented by the unit matrix  $\mathbf{1}$  in  $A$  and diagonal matrix  $d_0 = \text{diag}(1, \dots, 1, t_0)$ , for a fixed non square  $t_0 \in A$ .

Notice now that the values of  $S_{\underline{\psi}, \mathbb{Q}}$  at  $s = \mathbf{1}$  and  $s = a_0$  either coincide or differ by a sign. Indeed, since quadratic Gauss sums associated to an orthogonal sum of quadratic forms are just the product of the corresponding single Gauss sums, it may be readily checked that

$$\frac{S_{\underline{\psi}, \mathbb{Q}}(d_0)}{S_{\underline{\psi}, \mathbb{Q}}(1)} = \frac{S_{\underline{\psi}, t_0 Q_0}(1)}{S_{\underline{\psi}, Q_0}(1)}$$

so that our statement follows from the corresponding property for the classical rank 1 Gauss sum  $S_{\underline{\psi}, Q_0}$ , for which is well known or readily verified. Recalling that  $tQ' \simeq Q'$  for any non degenerate  $k$ -quadratic form  $Q'$  of even rank, we see then that

$$\frac{S_{\underline{\psi}, Q_0}(s)}{S_{\underline{\psi}, Q_0}(1)} = \alpha(s)$$

where  $\alpha(s) = 1$  for  $s$  in the orbit of  $\mathbf{1}$  and  $\alpha(s) = (-1)^m$  for  $s$  in the orbit of  $d_0$ , i.e.

$$(5.1) \quad \alpha(s) = (\text{sgn}(\det(s)))^m$$

for all  $s \in A^\times \cap A^{\text{sym}}$ . Notice that  $\alpha^2 = \mathbf{1}$  and  $\alpha = \alpha^{-1} = \bar{\alpha}$ . We put then

$$(5.2) \quad c = \frac{\alpha(-1)}{S_{\underline{\psi}, Q}}$$

Recall that we have  $tQ' \simeq Q'$  for any non degenerate quadratic form  $Q'$  of even rank over the finite field  $k$ .

We check now that  $\gamma$  satisfies 6.4. In fact

$$c\gamma(t, x) \sum_{y \in M} \chi(x, y) \gamma(t^{-1}, y) = c \sum_{y \in M} \underline{\psi}(tQ(x) + B(x, y) + t^{-1}Q(y))$$

Setting  $x = t^{-1}x'$  we get  $tQ(x) + B(x, y) + t^{-1}Q(y) = t^{-1}Q(x' + y)$ .

Then

$$\begin{aligned} c\gamma(t, x) \sum_{y \in M} \chi(x, y) \gamma(t^{-1}, y) &= c \sum_{y \in M} \underline{\psi}(t^{-1}Q(x' + y)) \\ &= c S_{\underline{\psi} \circ Q}(t^{-1}) \\ &= \alpha(-t^{-1}) \\ &= \alpha(-t). \end{aligned}$$

as desired.

Summing up, after having checked the required properties of our data, taking the character  $\alpha$  to be given by 5.1, for all  $s \in A^\times$  and then  $c$  to be given by 5.2 (see [11], Theorem 4.4 gives then a (true) Weil representation for  $SL_*^\varepsilon(2, A) = Sp(2n, k)$  for any non degenerate  $k$ -quadratic space  $(E_0, Q_0)$  and any choice of the non trivial additive character  $\psi$  of  $k$ . In this way we recover the representation constructed in [11] for even  $m$ , besides extending it to the odd  $m$  case.

## 6. Example 2: The truncated polynomial ring case

We give now an example of application of theorem 4.4 in the case of a non semi-simple involutive ring  $(A, *)$  with non-trivial nilpotent Jacobson radical.

Explicitly, we let  $k = \mathbb{F}_q$  be the finite field with  $q$  elements,  $q$  odd,  $m$  a positive integer. We set



$$A = A_m = k[x]/\langle x^m \rangle = \left\{ \sum_{i=0}^{m-1} a_i x^i : a_i \in k, x^m = 0 \right\}$$

and we denote by  $*$  the  $k$ -linear involution on  $A_m$  given by  $x \mapsto -x$ .

We will study here the group  $SL_*(2, A_m)$  for  $\varepsilon = -1$ , which will be denoted simply  $SL_*(2, A_m)$ . It is known that this group has a Bruhat presentation [5].

To this end let us consider the non-degenerate quadratic  $A$ -module  $(M, Q, B)$ , such that  $M = A_m$ ,  $Q : A_m \rightarrow A_m$  is given by  $Q(t) = a^*a$  and  $B : A_m \times A_m \rightarrow A_m$  is given by  $B(a, b) = a^*b + ab^*$ .

Then we have, for all  $a, b, t \in A_m$  :

- (i)  $Q(at) = t^*Q(a)t$ ;
- (ii)  $Q(a+b) = B(a, b) + Q(a) + Q(b)$ ;
- (iii)  $B(at, b) = B(a, bt^*)$ ;
- (iv)  $B(a, b) = B(b, a)^*$
- (vi)  $B(at, b) = t^*B(a, b)$ .

We denote by  $tr$  the linear form on  $A_m$  defined by  $tr\left(\sum_{i=0}^{m-1} a_i x^i\right) = a_{m-1}$ .

Then the form  $tr$  is  $k$ -linear and invariant under the involution  $*$ , i.e.,  $tr(a^*) = tr(a)$ , for any  $a \in A_m$ . Moreover the  $k$ -form  $tr \circ B$  is a non-degenerate symmetric bilinear form on  $A_m$ .

We fix be a non-trivial character  $\psi$  of  $k^+$  and we set  $\underline{\psi} = \psi \circ tr$ . We assume from now on that  $m$  be is odd.

The Gauss sum  $S_{\underline{\psi} \circ Q}$  associated to the character  $\underline{\psi}$  and the non-degenerate quadratic  $A_m$ -module  $(A_m, Q, B)$  is defined by

$$S_{\underline{\psi} \circ Q}(a) = \sum_{x \in A_m} \underline{\psi}(aQ(x)).$$

It is known [5] that the function  $\alpha$  from  $A_m^\times \cap A_m^{sym}$  to  $\mathbb{C}^\times$  given by  $\alpha(a) = \frac{S_{\underline{\psi} \circ Q}(a)}{S_{\underline{\psi} \circ Q}(1)}$  is the sign character of the group  $A_m^\times \cap A_m^{sym}$ .

Furthermore, we have

$$(6.1) \quad \alpha(tt^*) = 1 \quad (t \in A_m^\times)$$

$$(6.2) \quad (S_{\underline{\psi} \circ Q}(1))^2 = \alpha(-1)|A_m|$$

where  $|A_m|$  is the cardinality of the ring  $A_m$ .

Let us define the function  $\chi$  from  $A_m \times A_m$  to  $\mathbb{C}$  as  $\chi(a, b) = (\underline{\psi} \circ B)(a, b)$ , so  $\chi$  is a symmetric non degenerate biadditive form on  $A_m$  such that  $\chi(at, b) = \chi(a, bt^*)$  for all  $a, b, t \in A_m$ .

- (1)  $\chi(at, b) = \alpha(tt^*)\chi(a, bt^*)$  for  $a, b \in A_m$  and  $t \in A_m^\times$ .
- (2)  $\chi(b, b) = \chi(-a, b)$  for  $a, b \in A_m$ ,
- (3)  $\chi(a, b) = 1$  for any  $a \in A_m$  implies  $b = 0$ .

It follows from this and relations 6.1 and 6.2 that for the finite  $A_m$ -module  $M = A_m$  conditions 1a), 1b) and 1c) of subsection 4.1 hold.

We define now the function  $\gamma$  from  $A_m^{sym} \times A_m$  to  $\mathbb{C}$  as  $\gamma(b, x) = \underline{\psi}(bQ(x))$  ( $b \in A^{sym}, x \in A_m$ ) and we set  $c = \frac{\alpha(-1)}{S_{\underline{\psi} \circ Q}(1)}$ . Notice that  $c^2|A_m| = \alpha(-1)$ . Clearly the function  $\gamma$  is additive in the first variable, satisfies

$$\gamma(b, xt) = \gamma(tbt^*, x)$$

for  $t \in A_m^\times, x \in A_m, b \in A_m^{sym}$  and relates to  $\chi$  through

$$(6.3) \quad \gamma(b, x + y) = \gamma(b, x)\gamma(b, y)\chi(x, yb)$$

for any  $x, y \in A_m, b \in A_m^{sym}$ .

Finally

$$(6.4) \quad c\gamma(t, a) \sum_{d \in A_m} \chi(a, d)\gamma(t^{-1}, d) = \alpha(-t),$$

for any symmetric element  $t \in A_m^\times$ , where  $c \in \mathbb{C}^\times$  satisfies

$$(6.5) \quad c^2|A_m| = \alpha(-1).$$

The verification that  $\gamma$  satisfies 6.4 is completely analogous to the one in the preceding example.

Summing up, the above setup provides a Weil representation of  $SL_*^\varepsilon(2, A_m)$  in  $L^2(M)$ , according to theorem 4.4 (which is exactly the one constructed in [5]).

## 7. A first decomposition of the Weil representation

We give here a first decomposition of the Weil representation  $(L^2(M), \rho)$  of  $G$ , taking advantage of the fact that there is a group of intertwining operators that acts naturally in  $L^2(M)$ , to wit, the “unitary group”  $U(\gamma, \chi)$  of the pair  $(\gamma, \chi)$ .

**Definition 7.1.** We denote by  $U(\gamma, \chi)$  the group of all  $A$ -linear automorphisms  $\varphi$  of  $M$  such that

- (1)  $\gamma(b, \varphi(x)) = \gamma(b, x)$  for any  $b \in A^{sym}, x \in M$ ;
- (2)  $\chi(\varphi(x), \varphi(y)) = \chi(x, y)$  for any  $x, y \in M$ .

**Remark 7.2.** Condition (2) for  $\varphi$  in  $U(\gamma, \chi)$  is only necessary in the rather peculiar case where there are no  $\varepsilon$ -symmetric invertible elements in  $A$ , because if  $t \in A^\times \cap A^{sym}$ , then we have, for all  $x, y \in M$ ,

$$(7.1) \quad \chi(x, y) = \frac{\gamma(t, x + yt^{-1})}{\gamma(t, x)\gamma(t, yt^{-1})} = \frac{\gamma(t, \varphi(x) + \varphi(y)t^{-1})}{(\gamma(t, \varphi(x))\gamma(t, \varphi(y)t^{-1}))} = \chi(\varphi(x), \varphi(y)).$$

The next lemma addresses the converse question.

**Lemma 7.3.** *We can recover the function  $\gamma$  from  $\chi$  as follows.*

$$(7.2) \quad \gamma(t, x) = \chi(x2^{-1}t^*, x) \quad (t \in A^\times \cap A^{sym}, x \in M)$$

PROOF. To prove relation 7.2 we calculate  $\gamma(r, xs + xs)$ , for  $x \in M, r, s \in A^\times \cap A^{sym}$ , in two different ways:

$$\begin{aligned} \gamma(r, xs + xs) &= \gamma(2srs^*, x)\chi(xs, xsr), \\ \gamma(r, xs + xs) &= \gamma(2sr2s^*, x) = \gamma(4srs^*, x). \end{aligned}$$

Therefore

$$\gamma(2srs^*, x) = \chi(xs, xsr)$$

and choosing  $r, s \in A^\times \cap A^{sym}$  such that  $sr = 1$  we get

$$\gamma(2s^*, x) = \chi(xs, x)$$

Putting  $2s^* = t$ , we finally get  $\gamma(t, x) = \chi(x2^{-1}t^*, x)$ .

□

**Proposition 7.4.** *The group  $\Gamma = U(\gamma, \chi)$  acts naturally on  $L^2(M)$  by  $(\varphi.f)(x) = f(\varphi^{-1}(m))$ . This action commutes with the Weil representation  $\rho$  of  $G = SL_*^\varepsilon(2, A)$*

PROOF. Using the explicit definition in terms of  $\chi$  and  $\gamma$ , of the Weil operators  $\rho(g)$  for our generators  $g$  of  $G$ , besides the fact that all  $\varphi \in \Gamma$  are  $A$ -linear, one readily checks that the natural action of  $\Gamma$  commutes with them. □

**Definition 7.5.** Let  $(\pi, V)$  be an irreducible representation of  $\Gamma = U(\gamma, \chi)$ . Denote by  $(L_V^2(M)[\pi], \rho)$  the representation of  $G$  in the space  $L_V^2(M)[\pi]$  consisting of all  $V$ -valued functions  $f$  on  $M$  such that  $f(\varphi(x)) = \pi(\varphi)(f(x))$ , whose action is the action of the Weil representation  $\rho$  of  $SL_*^\varepsilon(2, A)$ .

Now the well known description of isotypical components for the action of one group that intertwines with the action of another group in the same space (see [11], for instance) becomes:

**Proposition 7.6.** *The Weil  $(L^2(M), \rho)$  of  $G$  decomposes as the direct sum of the isotypical components of the natural representation of  $\Gamma$  in  $L^2(M)$  as follows:*

$$W = \bigoplus_{\pi \in \widehat{\Gamma}} \pi^\vee \otimes L_V^2(M)[\pi],$$

where  $\pi^\vee$  denotes the contragredient of the irreducible representation  $\pi \in$  of  $\widehat{\Gamma}$ .

The irreducibility of the representations  $(L_V^2(M)[\pi], \rho)$  of  $G$  constructed in this way, remains an open problem in general (see [11] for a complete answer for  $G = Sp(4, \mathbb{F}_q)$ ). We expect to address this question in other cases elsewhere.

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